STANLEY DECOMPOSITIONS OF SQUAREFREE MODULES AND ALEXANDER DUALITY

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ABSTRACT. In this paper we study how prime filtrations and squarefree Stanley decompositions of squarefree modules over the polynomial ring and the exterior algebra behave with respect to Alexander duality.

INTRODUCTION

Let K be a field and $S = K[x_1,\dots,x_n]$ the polynomial ring in n variables. The ring S is naturally $\mathbb{N}^n$-graded. Yanagawa [27] introduced squarefree S-modules which generalizes the concept of Stanley–Reisner rings. A finitely generated $\mathbb{N}^n$-graded S-module $M = \bigoplus_{\mathbf{a}\in\mathbb{N}^n} M_{\mathbf{a}}$ is *squarefree* if the multiplication map $M_{\mathbf{a}} \to M_{\mathbf{a}+\varepsilon_i}$, $m \mapsto mx_i$, is bijective for all $\mathbf{a}\in\mathbb{N}^n$ and all $i \in \operatorname{supp}(\mathbf{a})$. Römer defined in [18] the Alexander dual $M^\vee$ for a squarefree S-module M. The definition refers to exterior algebras. Let E be the exterior algebra over an n-dimensional K-vector space V. A finitely generated $\mathbb{N}^n$-graded E-module $N = \bigoplus_{\mathbf{a}\in\mathbb{N}^n} N_{\mathbf{a}}$ is called *squarefree* if it has only squarefree components. By [18, Corollary 1.6] the category of squarefree S-modules is equivalent to the category of squarefree E-modules. For an $\mathbb{N}^n$-graded E-module N the E-dual of N is the graded dual $N^\vee = \operatorname{Hom}_E(N,E)$. Let M be a squarefree S-module and N its corresponding squarefree E-module. Then $M^\vee$ is defined to be the squarefree S-module corresponding to $N^\vee$. In the first section of this paper we recall some basic notion and definitions about squarefree S-modules and E-modules. In Section 2 we study prime filtrations of squarefree S-modules and E-modules. As a main result of this section we prove that for a squarefree S-module M there exists a chain $0 \subset M_1 \subset \cdots \subset M_r = M$ of squarefree submodules of M with $M_i/M_{i-1} \cong S/P_{F_i}(-G_i)$ if and only if there exists a chain $0 \subset L_1 \subset \cdots \subset L_r = M^\vee$ of squarefree submodules of $M^\vee$ with $L_i/L_{i-1} \cong S/P_{G_i}(-F_i)$, see Theorem 2.3. For proving this, in Proposition 2.2 we show that the corresponding result is true for squarefree E-modules. In Corollary 2.4 we show explicitly how the prime filtration of $M^\vee$ is obtained form that of M, in the special case that $M = J/I$, where $I \subset J$ are squarefree monomial ideals.

In last section we study Stanley decompositions of finitely generated $\mathbb{Z}^n$-graded S-modules. Let $m \in M$ be a homogeneous element and $Z \subset \{x_1,\dots,x_n\} = X$. We denote by $mK[Z]$ the K-subspace of M generated by all homogeneous elements of the form mu, where u is a monomial in $K[Z]$. The K-subspace $mK[Z]$ is called a *Stanley space of dimension* $|Z|$ if $mu \neq 0$ for all nonzero monomial $u \in K[Z]$. Here $|Z|$ denote the cardinality of Z. A homogeneous element $m \in M$ is called squarefree if $\deg(m) = (a_1,\dots,a_n) \in \{0,1\}^n$. The Stanley space $mK[Z]$ is called *squarefree* if m is a squarefree homogeneous element and $\operatorname{supp}(\deg(m)) = \{j\colon a_j \neq 0\} \subset \{i\colon x_i \in Z\}$.

A decomposition $\mathscr{D}$ of M as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of M. The Stanley decomposition $\mathscr{D}$ of M is called *squarefree Stanley decomposition* if all Stanley spaces in $\mathscr{D}$ are squarefree Stanley spaces. In Proposition 3.2 we show that the R-module M has a squarefree Stanley decomposition if and only if M is squarefree R-module. The minimal dimension of a Stanley space in the decomposition $\mathscr{D}$ is called the *Stanley depth* of $\mathscr{D}$, denoted by

sdepth($\mathscr{D}$). We set

$$\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathscr{D})\colon \mathscr{D} \text{ is a Stanley decomposition of } M\},$$

and call this number the *Stanley depth* of M. For a squarefree module M we denote by

$$\operatorname{sqdepth}(M) = \max\{\operatorname{sdepth}(\mathscr{D})\colon \mathscr{D} \text{ is a squarefree Stanley decomposition of } M\}$$

the *squarefree Stanley depth* of M. If M is squarefree, then $\operatorname{sqdepth}(M) = \operatorname{sdepth}(M)$, see Theorem 3.4.

As a main result of this section we show that a squarefree S-module M has a squarefree Stanley decomposition $M = \bigoplus_{i=1}^{t} m_i K[Z_i]$ if and only if there exist a squarefree Stanley decomposition $M^\vee = \bigoplus_{i=1}^{t} v_i K[W_i]$ of $M^\vee$ with $\operatorname{supp}(v_i) = [n] \setminus \{j\colon x_j \in Z_i\}$ and $W_i = \{x_j\colon j \in [n] \setminus \operatorname{supp}(m_i)\}$, see Theorem 3.7. To prove this we show in Proposition 3.5 that the correspomding result is true for squarefree E-modules. As corollaries of Theorem 3.7 we show that Stanley's conjecture on Stanley decompositions holds for a squarefree S-module M if and only if $M^\vee$ has a Stanley decomposition $M^\vee = \bigoplus_{i=1}^{t} v_i K[W_i]$ with $|v_i| \le \operatorname{reg}(M^\vee)$ for all i, see Corollary 3.8, and Stanley's conjecture on partitionable simplicial complexes holds for a Cohen–Macaulay simplicial complex Δ if and only if $I_{\Delta^\vee}$ has a Stanley decomposition $I_{\Delta^\vee} = \bigoplus_{i=1}^{t} u_i K[Z_i]$ such that $\{u_i, \ldots, u_t\} = G(I_{\Delta^\vee})$.

Due to these facts we conjecture (Conjecture 3.10) that any $\mathbb{Z}^n$-graded S-module M has a Stanley decomposition $M = \bigoplus_{i=1}^{t} m_i K[Z_i]$ with $|m_i| \le \operatorname{reg}(M)$. In some cases we can show that this conjecture holds.

Acknowledgements. I am grateful to Professor Jürgen Herzog for his suggestion to consider Stanley decompositions of a squarefree S-module and their Alexander dual, and for useful discussions and comments.

1. SQUAREFREE MODULES AND ALEXANDER DUAL

We fix some notation and recall some definitions. For $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we say $\mathbf{a}$ is squarefree if $a_i = 0$ or $a_i = 1$ for $i = 1, \ldots, n$. We set $\operatorname{supp}(\mathbf{a}) = \{i\colon a_i \neq 0\} \subset [n] = \{1, \ldots, n\}$ and $|\mathbf{a}| = \sum_{i=1}^{n} a_i$. Occasionally we identify a squarefree vector $\mathbf{a}$ with $\operatorname{supp}(\mathbf{a})$. Let $\varepsilon_i = (0\ldots, 1, \ldots, 0) \in \mathbb{N}^n$ be the vector with 1 at the i-th position. Let $M = \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} M_{\mathbf{a}}$ be an $\mathbb{Z}^n$-graded K-vector space. For simplicity set $\operatorname{supp}(m) = \operatorname{supp}(\deg m)$ and $|m| = |\deg m|$ for any homogeneous element $m \in M$. A homogeneous element $m \in M$ is called *squarefree* if $\deg m \in \{0,1\}^n$.

Let K be a field and $S = K[x_1, \ldots, x_n]$ the symmetric algebra over K. Consider the natural $\mathbb{N}^n$-grading on S. For a monomial $x_1^{a_1} \cdots x_n^{a_n}$ with $\mathbf{a} = (a_1, \ldots, a_n)$ we set $x^{\mathbf{a}}$, and for $F \subset [n]$ we denote $x_F = \prod_{j \in F} x_j$.

Let V be an n-dimensional K-vector space with basis $e_1, \ldots, e_n$. We denote by $E = K\langle e_1, \ldots, e_n\rangle$ the exterior algebra over V. The algebra E is a naturally $\mathbb{N}^n$-graded K-algebra with $\deg e_i = \varepsilon_i$. Let $F = \{j_1 < j_2 < \ldots < j_k\} \subset [n]$. Then $e_F = e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_k}$ is called a monomial in E. It is easy to see that the elements e_F, with $F \subset [n]$ form a K-basis of E. Here we set $e_F = 1$, if $F = \emptyset$. For any $\mathbf{a} \in \mathbb{N}^n$ we set $e_{\mathbf{a}} = e_{\operatorname{supp}(\mathbf{a})}$.

A finite dimensional K-vector space M is called an $\mathbb{Z}^n$-graded E-module, if

(i) $M = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} M_{\mathbf{a}}$ is a direct sum of K-vector spaces $M_{\mathbf{a}}$;
(ii) M is an $(E-E)$-bimodule;
(iii) for all vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{Z}^n$ and all $f \in E_{\mathbf{a}}$ and $m \in M_{\mathbf{b}}$ one has $fm \in M_{\mathbf{a}+\mathbf{b}}$ and $fm = (-1)^{|\mathbf{a}||\mathbf{b}|} mf$.

A simplicial complex Δ is a collection of subset of $[n] = \{1, \ldots, n\}$ such that whenever $F \in \Delta$ and $G \subset F$, then $G \in \Delta$. Further we denote by $\Delta^\vee = \{F\colon F^c \notin \Delta\}$ the Alexander dual of Δ, where $F^c = [n] \setminus F$. Then $K[\Delta] = S/I_\Delta$ is called the Stanley–Reisner ring, where $I_\Delta = (x_{i_1} \cdots x_{i_k}\colon \{i_1, \ldots, i_k\} \notin$

Δ), and $K\{\Delta\} = E/J_\Delta$ is called the exterior face ring of Δ, where $J_\Delta = (e_{i_1} \wedge \cdots \wedge e_{i_k} : \{i_1, \ldots, i_k\} \notin \Delta)$.

The following definition is due to Yanagawa [27].

Definition 1.1. *A finitely generated $\mathbb{N}^n$-graded S-module $M = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} M_\mathbf{a}$ is* squarefree *if the multiplication map $M_\mathbf{a} \to M_{\mathbf{a}+\varepsilon_i}$, $m \mapsto mx_i$, is bijective for all $\mathbf{a} \in \mathbb{N}^n$ and all $i \in \mathrm{supp}(\mathbf{a})$.*

For examples the Stanley-Reisner ring $K[\Delta]$ of a simplicial complex Δ is a squarefree S-module. If $I \subset J$ are squarefree monomial ideals, then I, S/I and J/I are squarefree S-modules. The following example shows that the factor module J/I may be a squarefree $\mathbb{N}^n$-graded S-module, even though I and J are not squarefree monomial ideals.

Example 1.2. Let $I = (x^2, xy) \subset J = (x^2, xy, yz)$ be monomial ideals in $K[x, y, z]$. Then an element $u \in J \setminus I$ if and only if $u = (yz)v$ for some $v \in K[y, z]$. Hence J/I is a squarefree $\mathbb{N}^n$-graded S-module. But if we choose $I' = (x^2, yz) \subset J = (x^2, xy, yz) \subset K[x, y, z]$, then $xy \in J \setminus I$ and $x(xy) = x^2y \in I'$. Therefore J/I' is not a squarefree $\mathbb{N}^n$-graded S-module.

Since $\dim_K (J/I)_\mathbf{a} \leq 1$ for all $\mathbf{a} \in \mathbb{N}^n$, the $\mathbb{N}^n$-graded S-module J/I is squarefree if and only if the multiplication map

$$(J/I)_\mathbf{a} \to (J/I)_{\mathbf{a}+\varepsilon_i},\ m \to x_i m$$

is injective for all $i \in \mathrm{supp}(m)$ and all $\mathbf{a} \in \mathbb{N}^n$.

Remark 1.3. Let $I \subset J \subset S$ be two monomial ideals. The $\mathbb{N}^n$-graded S-module J/I is squarefree if and only all minimal monomial generators of J/I are squarefree monomials and $\mathrm{supp}(u) \not\subset \mathrm{supp}(m)$ for all $m \in J \setminus I$ and all $u \in G(I)$ where $G(I)$ denote the set of minimal monomial generators of I. Indeed let J/I be a squarefree S-module and one of the minimal generators of J/I is not sqaurefree, say $m \in J \setminus I$. We may assume that $x_1^2 \mid m$ and $\deg(m) = \mathbf{a}$. Then $m' = m/x_1 \in (J/I)_{\mathbf{a}-\varepsilon_i}$ is a zero element and $1 \in \mathrm{supp}(m')$ but $m = x_1 m' \in (J/I)_\mathbf{a}$ is a nonzero element, a contradiction. Also if there exists a monomial $m \in J \setminus I$ and there exists a monomial $u \in G(I)$ such that $\mathrm{supp}(u) \subset \mathrm{supp}(m)$. Then in this case one can find a minimal monoial $m' = mx^\mathbf{a}$ (with respect to divisibility) such that $\mathrm{supp}(\mathbf{a}) \subset \mathrm{supp}(m)$, $u \mid m'$ and $m'/x_i \notin I$ for some $i \in \mathrm{supp}(\mathbf{a})$, again a contradiction.

For the converse assume that J/I is minimally generated by squarefree monomials in $J \setminus I$ and $\mathrm{supp}(u) \not\subset \mathrm{supp}(m)$ for all $m \in J \setminus I$ and for all $u \in G(I)$. Let $m \in S$ be a monomial and $i \in \mathrm{supp}(m)$. Since the minimal monomial generators of J/I are squarefree, if $m \notin J$, then $x_i m \notin J$ or $x_i m \in J \cap I$. Hence in this case the multiplication map $m \to x_i m$ is injective. In the case that if $m \in J \setminus I$, then $x_i m \notin I$. Otherwise there must exist a $u \in G(I)$ such that $u \mid x_i m$. Therefore $\mathrm{supp}(u) \subset \mathrm{supp}(x_i m) = \mathrm{supp}(m)$ which is a contradiction.

Yanagawa [27, Lemma 2.3] proved that if M and M' are squarefree S-modules and $\varphi : M \to M'$ is a $\mathbb{N}^n$-homogeneous homomorphism, then $\mathrm{Ker}\,\varphi$ and $\mathrm{Coker}\,\varphi$ are again squarefree S-modules. This implies that each syzygy module $\mathrm{Syz}_i(M)$ in a multigraded minimal free S-resolution $F_\bullet$ of M is squarefree.

It is easy to see that if M is a squarefree S-module, then $\dim_K M_\mathbf{a} = \dim_K M_{\mathrm{supp}(\mathbf{a})}$ for any $\mathbf{a} \in \mathbb{N}^n$, and M is generated by its squarefree parts $\{M_F : F \subset [n]\}$.

Next we recall the following definition which is due to T. Römer [18].

Definition 1.4. *A finitely generated $\mathbb{N}^n$-graded E-module $N = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} N_\mathbf{a}$ is called squarefree if it has only squarefree components.*

For example the exterior face ring $K\{\Delta\}$ of a simplicial complex Δ is a squarefree E-module.

We denote by $SQ(S)$ the abelian category of the squarefree S-modules, where the morphisms are the $\mathbb{N}^n$-graded homogeneous homomorphisms and denote by $SQ(E)$ the abelian category of squarefree E-modules, where the morphisms are the $\mathbb{N}^n$-graded homogeneous homomorphisms. Römer [18, Corollary 1.6] proved that there are two exact additive covariant functors

$$\mathbf{F}\colon SQ(S)\mapsto SQ(E),\ M\mapsto \mathbf{F}(M) \quad \text{and} \quad \mathbf{G}\colon SQ(E)\mapsto SQ(S),\ N\mapsto \mathbf{G}(N)$$

of abelian categories such that $(\mathbf{F}\circ\mathbf{G})(N)=N$ and $(\mathbf{G}\circ\mathbf{F})(M)=M$. Hence the categories $SQ(S)$ and $SQ(E)$ are equivalent. Let $M\in SQ(S)$. By the construction of $N=\mathbf{F}(M)$ given in [1] and [18], each minimal homogeneous system of generators $m_1,\dots,m_t$ of M corresponds to a homogeneous minimal system of generators $n_1,\dots,n_t$ of $N=\mathbf{F}(M)$, and for all $F\subset[n]$ we have an isomorphism of K-vector spaces $\theta_F\colon M_F\to\mathbf{F}(M)_F$. This isomorphism is described as follows: an element $m\in M_F$ can be written as $m=\sum a_i m_i x_{F_i}$, where $a_i\in K$ and where F is the disjoint union of F_i and $\deg(m_i)=G_i$ for each i. Then

$$\theta_F(m)=\sum(-1)^{\sigma(G_i,F_i)}a_i n_i e_{F_i}, \tag{1}$$

where $\sigma(G_i,F_i)=|\{(r,s)\colon r\in G_i,\ s\in F_i,\ r>s\}|$. The definition of θ_F does not depend on the particular presentation of m as a homogeneous linear combination of the m_i. In particular, we have that $\theta_{G_i}(m_i)=n_i$ for all i.

We set $M_{\text{sq}}=\bigoplus_F M_F$ and define the isomorphism of graded K-vector spaces $\theta\colon M_{\text{sq}}\to N$ by requiring that $\theta(m)=\theta_F(m)$ for all $m\in M_F$. Now Formula (1) can be extended as follows:

Lemma 1.5. *Let m be a squarefree element of M with $\operatorname{supp}(m)=F$, and let $m=\sum_i a_i w_i x_{L_i}$ with $a_i\in K$ and w_i squarefree with $\operatorname{supp}(w_i)=F_i$ such that F is the disjoint union of F_i and L_i for all i. Then*

$$\theta(m)=\sum a_i(-1)^{\sigma(F_i,L_i)}\theta(w_i)e_{L_i}.$$

Proof. Let $m_1,\dots,m_t$ be a minimal homogeneous system of generators of M and let $n_1,\dots,n_t$ be the corresponding minimal homogeneous system of generators of N with $\theta(m_i)=n_i$. Let $w_i=\sum b_{ij}m_{ij}x_{H_{ij}}$ where $b_{ij}\in K$ and where F_i is a disjoint union of $G_{ij}=\operatorname{supp}(m_{ij})$ and H_{ij} for all ij. Then

$$\theta(m)=\theta(\sum_i a_i(\sum_j b_{ij}m_{ij}x_{H_{ij}})x_{L_i}=\theta(\sum_i\sum_j a_i b_{ij}m_{ij}x_{H_{ij}\cup L_i})=\sum_i\sum_j(-1)^{\sigma(G_{ij},H_{ij}\cup L_i)}n_{ij}e_{H_{ij}\cup L_i}.$$

On the other hand

$$\begin{aligned}
\sum a_i(-1)^{\sigma(F_i,L_i)}\theta(w_i)e_{L_i} &= \sum_i\sum_j(-1)^{\sigma(G_{ij}\cup H_{ij},L_i)}(-1)^{\sigma(G_{ij},H_{ij})}a_i b_{ij}n_{ij}e_{H_{ij}}e_{L_i}\\
&= \sum_i\sum_j(-1)^{\sigma(G_{ij},L_i)}(-1)^{\sigma(H_{ij},L_i)}(-1)^{\sigma(G_{ij},H_{ij})}(-1)^{\sigma(H_{ij},L_i)}a_i b_j n_{ij}e_{H_{ij}\cup L_i}\\
&= \sum_i\sum_{ij}(-1)^{\sigma(G_{ij},H_{ij}\cup L_i)}n_{ij}e_{H_{ij}\cup L_i}=\theta(m).
\end{aligned}$$

□

Let W be an $\mathbb{Z}^n$-graded K-vector space. Then $W^*=\operatorname{Hom}_K(W,K(-\mathbf{1}))$ is again a $\mathbb{Z}^n$-graded K-vector space with the graded components

$$(W^*)_{\mathbf{a}}=\operatorname{Hom}_K(W_{\mathbf{1}-\mathbf{a}},K) \text{ for all } \mathbf{a}\in\mathbb{Z}^n.$$

Here $\mathbf{1}=(1,\dots,1)$. Note that if W is an $\mathbb{Z}^n$-graded E-module, then W^* is also a $\mathbb{Z}^n$-graded E-module. Furthermore if W is a squarefree E-module, then W^* is again a squarefree E-module.

In the category of squarefree E-modules the graded E-dual is defined to be $N^\vee=\operatorname{Hom}_E(N,E)$. Observe that $()^\vee$ is an exact contravariant functor, see [2, 5.1(a)]. Let $\varphi\in N^\vee$ and $n\in N$. Then

$\varphi(n)=\sum_{F\subseteq[n]}\varphi_F(n)e_F$ with $\varphi_F(n)\in K$ for all $F\subseteq[n]$. Therefore for each $F\subseteq[n]$ we obtain a K-linear map $\varphi_F: N\to K$.

The following theorem is important for the main result of this paper.

Theorem 1.6. [9] *Let N be a $\mathbb{Z}^n$-graded E-module. The map $\eta: N^\vee\to N^*$, $\varphi\to\varphi_{[n]}$ is a functorial isomorphism of $\mathbb{Z}^n$-graded E-modules. In particular if N is squarefree E-module, then $N^\vee$ is again squarefree and η is a functorial isomorphism of squarefree E-modules.*

In [18], the Alexander dual of a squarefree S-module is defined as follows:

Definition 1.7. *Let $M\in SQ(S)$. Then $M^\vee=\mathbf{G}(\mathbf{F}(M)^\vee)$ is called the Alexander dual of M.*

Note that

$$SQ(S)\to SQ(S),\quad M\to M^\vee$$

is a contravariant exact functor.

For example if $I\subset J$ are squarefree monomial ideals in S. Let Δ and Γ be simplicial complexes with $I=I_\Delta$ and $J=I_\Gamma$. Then J/I is a squarefree S-module and $(J/I)^\vee=I_{\Delta^\vee}/I_{\Gamma^\vee}$. In particular if Δ is a simplicial complex on the vertex set $[n]$ and I_Δ its Stanley-Reisner ideal, then $(S/I_\Delta)^\vee=I_{\Delta^\vee}$ and $(I_\Delta)^\vee=S/I_{\Delta^\vee}$.

2. Prime filtrations and Alexander duality

Let $S=K[x_1,\dots,x_n]$ be the polynomial ring in n variables over a field K and M a finitely generated $\mathbb{Z}^n$-graded S-module. It is known that the associated prime ideals of M are monomial ideals, and any monomial prime ideal is of the form $P_F=(x_i\colon i\in F)$ for some $F\subset[n]$. A chain $0=M_0\subset M_1\subset\ldots\subset M_r=M$ of $\mathbb{Z}^n$-graded submodules of M such that $M_i/M_{i-1}\cong S/P_{F_i}(-G_i)$ is called a prime filtration of M. If M is a finitely generated $\mathbb{Z}^n$-graded S-module, then a prime filtration of M always exists, see [15, Theorem 6.4].

We shall need the following

Lemma 2.1. *Let $M\subset M'$ be two squarefree S-modules and $N\subset N'$ be two squarefree E-modules.*

(a) *If $M'/M\cong S/P_F(-G)$, then $G\cap F=\emptyset$;*
(b) *We have $M'/M\cong S/P_F(-G)$ if and only if $\mathbf{F}(M')/\mathbf{F}(M)\cong E/P_{F\cup G}(-G)$, where $P_{F\cup G}=(e_j\colon j\in F\cup G)$;*
(c) *We have $N'/N\cong E/P_{F\cup G}(-G)$ if and only if $\mathbf{G}(N')/\mathbf{G}(N)\cong S/P_F(-G)$.*

Proof. (a) Suppose $G\cap F\neq\emptyset$. Let $i\in G\cap F$ and let f the homogeneous generator of M'/M. Since M'/M is squarefree, and since $\deg f=G$ it it follows that $x_if\neq 0$, a contradiction.

(b) Since $\mathbf{F}$ is an exact functor it suffices to show that $\mathbf{F}(S/P_F(-G))=E/P_{F\cup G}(-G)$. But this follows immediately from the Aramova-Avramov-Herzog complex [1, Theorem 1.3] by which Römer defined the functor $\mathbf{F}$ in [18].

(c) follows form (b) by using the fact that the functors $\mathbf{F}$ and $\mathbf{G}$ are inverse to each other. $\square$

Applying this lemma we get the following short exact sequence

$$0\to\mathbf{F}(M)\to\mathbf{F}(M')\to E/P_{F\cup G}(-G)\to 0.$$

Since $\operatorname{Hom}_E(-,E)$ is an contravariant exact functor, from the above short exact sequence we obtain the short exact sequence

$$0\to\operatorname{Hom}_E(E/P_{F\cup G}(-G),E)\to\mathbf{F}(M')^\vee\to\mathbf{F}(M)^\vee\to 0.$$

On the other hand $\operatorname{Hom}_E(E/P_{F\cup G}(-G),E)=\operatorname{Hom}_E(E/P_{F\cup G},E)(G)$. Since

$$\operatorname{Hom}_E(E/P_{F\cup G},E)=0:_E P_{F\cup G}=(e_{F\cup G})\cong E/P_{F\cup G}(-F-G),$$

one has $\mathrm{Hom}_E(E/P_{F\cup G}(-G),E) \cong E/P_{F\cup G}(-F)$.

We conclude that the natural map

$$\alpha\colon \mathbf{F}(M')^\vee \to \mathbf{F}(M)^\vee$$

is an epimorphism with $\mathrm{Ker}(\alpha) \cong E/P_{F\cup G}(-F)$.

Proposition 2.2. *Let N be a squarefree E-module and $N^\vee$ its E-dual. Then there exists a chain $0 \subset N_1 \subset \ldots \subset N_t = N$ of squarefree submodules of N with $N_i/N_{i-1} \cong E/P_{F_i\cup G_i}(-G_i)$ if and only if there exists a chain $0 \subset H_1 \subset \ldots \subset H_t = N^\vee$ of squarefree submodule of $N^\vee$ with $H_i/H_{i-1} \cong E/P_{F_i\cup G_i}(-F_i)$.*

Proof. It is enough to prove one direction of the assertion, because $(N^\vee)^\vee = N$. Let $0 = N_0 \subset N_1 \subset \ldots \subset N_t = N$ be a chain of squarefree E-modules with $N_i/N_{i-1} \cong E/P_{F_i\cup G_i}(-G_i)$. From the observation above we see that for each i there is an epimorphism $\alpha_i\colon N^\vee_{t-i+1} \to N^\vee_{t-i}$ with $\mathrm{Ker}\,\alpha_i \cong E/P_{F_i\cup G_i}(-F_i)$.

Let $\beta_i\colon N^\vee \to N^\vee_{t-i}$ be the epimorphism which is defined by $\beta_i = \alpha_i \circ \alpha_{i-1} \circ \cdots \circ \alpha_1$. Then

$$0 \subset \mathrm{Ker}\,\beta_1 \subset \cdots \subset \mathrm{Ker}\,\beta_t = N^\vee$$

is a filtration of $N^\vee$ by squarefree E-modules. We only need to show that $\mathrm{Ker}\,\beta_i/\mathrm{Ker}\,\beta_{i-1} \cong \mathrm{Ker}\,\alpha_i$. This follows from the Snake Lemma applied to the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}\,\beta_{i-1} & \xrightarrow{\iota_1} & N^\vee & \xrightarrow{\beta_{i-1}} & N^\vee_{t-i+1} & \longrightarrow & 0 \\
 & & {\scriptstyle\iota_2}\big\downarrow & & {\scriptstyle\mathrm{id}}\big\downarrow & & {\scriptstyle\alpha_i}\big\downarrow & & \\
0 & \longrightarrow & \mathrm{Ker}\,\beta_i & \xrightarrow{\iota_3} & N^\vee & \xrightarrow{\beta_i} & N^\vee_i & \longrightarrow & 0
\end{array}$$

with exact rows, where the ι_j are inclusion maps. □

Now we can prove the corresponding result for squarefree S-modules.

Theorem 2.3. *Let M be a squarefree S-module and $M^\vee$ its Alexander dual. Then there exists a chain $0 \subset M_1 \subset \cdots \subset M_r = M$ of squarefree submodules of M with $M_i/M_{i-1} \cong S/P_{F_i}(-G_i)$ if and only if there exists a chain $0 \subset L_1 \subset \cdots \subset L_r = M^\vee$ of squarefree submodules of $M^\vee$ with $L_i/L_{i-1} \cong S/P_{G_i}(-F_i)$.*

Proof. Again it is enough to prove one direction of the assertion, because $(M^\vee)^\vee = M$. From the given chain of submodules of M we get a chain

$$0 \subset \mathbf{F}(M_1) \subset \cdots \subset \mathbf{F}(M_r) = \mathbf{F}(M)$$

of squarefree E-modules with $\mathbf{F}(M_i)/\mathbf{F}(M_{i-1}) \cong E/P_{F_i\cup G_i}(-G_i)$, see Lemma 2.1(b). Therefore by Proposition 2.2 there exists a chain $0 \subset N_1 \subset \cdots \subset N_{r-1} \subset N_r = (\mathbf{F}(M))^\vee$ of squarefree E-modules with $N_i/N_{i-1} \cong E/P_{F_i\cup G_i}(-F_i)$. This chain of squarefree E-modules induces the chain

$$0 \subset \mathbf{G}(N_1) \subset \cdots \subset \mathbf{G}(N_{r-1}) \subset \mathbf{G}(N_r) = \mathbf{G}(\mathbf{F}(M))^\vee) = M^\vee$$

of squarefree S-modules with $\mathbf{G}(N_i)/\mathbf{G}(N_{i-1}) \cong S/P_{G_i}(-F_i)$, see Lemma 2.1(c). □

We now explain what Theorem 2.3 means in the special case that $M = J/I$ where $I \subset J \subset S$ are squarefree monomial ideals. To this end we introduce the following notation: let $I \subset S$ be a squarefree monomial ideal and Δ be the simplicial complex such that $I = I_\Delta$. We set $\tilde{I} = I_{\Delta^\vee}$. Then $\tilde{\tilde{I}} = I$ since $(\Delta^\vee)^\vee = \Delta$, and if $I \subset J$ are two squarefree monomial ideals, then $\tilde{J} \subset \tilde{I}$ and $(J/I)^\vee = \tilde{I}/\tilde{J}$.

Corollary 2.4. *Let $I \subset J$ be a squarefree monomial ideals. The following conditions are equivalent:*

(a) *$I = I_0 \subset I_1 \subset \ldots \subset I_{r-1} \subset I_r = J$ is an $\mathbb{N}^n$-graded prime filtration of J/I with $I_i/I_{i-1} \cong S/P_{F_i}(-G_i)$.*
(b) *$\tilde{J} = \tilde{I}_r \subset \tilde{I}_{r-1} \subset \ldots \subset \tilde{I}_1 \subset \tilde{I}_0 = \tilde{I}$ is an $\mathbb{N}^n$-graded prime filtration of $\tilde{I}/\tilde{J} = (J/I)^\vee$ with $\tilde{I}_{i-1}/\tilde{I}_i \cong S/P_{G_i}(-F_i)$.*

Proof. It is enough to prove the implication (a) $\Rightarrow$ (b), because $\tilde{\tilde{L}} = L$ for any squarefree monomial ideal L. For the proof we may assume that $r = 1$, in other words $J/I \cong S/P_F(-G)$. In this situation $\tilde{I}/\tilde{J} = (J/I)^\vee \cong S/P_G(-F)$, by Theorem 2.3. $\square$

3. Stanley decompositions and Alexander duality

In [21, Conjecture 5.1] Stanley conjectured the following: let R be a finitely generated $\mathbb{N}^n$-graded K-algebra (where $R_0 = K$ as usual), and let M be a finitely generated $\mathbb{Z}^n$-graded R-module. Then there exist finitely many subalgebras $S_1, \ldots, S_t$ of R, each generated by algebraically independent $\mathbb{N}^n$-homogeneous elements of R, and there exist $\mathbb{Z}^n$-homogeneous elements $m_1, \ldots, m_t$ of M, such that

$$M = \bigoplus_{i=1}^{t} m_i S_i$$

where $\dim S_i \geq \operatorname{depth} M$ for all i, and where $m_i S_i$ is a free S_i-module (of rank one). Moreover, if K is infinite and under a given specialization to an $\mathbb{N}$-grading R is generated by R_1, then we can choose the ($\mathbb{N}^n$-homogeneous) generators of each S_i to lie in R_1.

Stanley's conjecture has been studied in several articles, see for examples [5], [6], [20], [13], [3], [4], [17] and [24].

We consider this conjecture in the case that M is a finitely generated $\mathbb{Z}^n$-graded S-module, where $S = K[x_1, \ldots, x_n]$ is the polynomial ring in n variables. Let $m \in M$ be a homogeneous element and $Z \subset \{x_1, \ldots, x_n\} = X$. We denote by $mK[Z]$ the K-subspace of M generated by all homogeneous elements of the form mu, where u is a monomial in $K[Z]$. The K-subspace $mK[Z]$ is called a *Stanley space of dimension* $|Z|$ if $mu \neq 0$ for any nonzero monomial $u \in K[Z]$. According to [13] the Stanley space $mK[Z]$ is called *squarefree* if m is a squarefree homogeneous element and $\operatorname{supp}(m) \subset \operatorname{supp}(Z) = \{i \colon x_i \in Z\}$.

A decomposition $\mathscr{D}$ of M as a finite direct sum of Stanley spaces is called a *Stanley decomposition* of M. The Stanley decomposition $\mathscr{D}$ of M is called a *squarefree Stanley decomposition* if all Stanley spaces in $\mathscr{D}$ are squarefree Stanley spaces. The minimal dimension of a Stanley space in the decomposition $\mathscr{D}$ is called the *Stanley depth* of $\mathscr{D}$, denoted $\operatorname{sdepth}(\mathscr{D})$. We set

$$\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathscr{D}) \colon \mathscr{D} \text{ is a Stanley decomposition of } M\},$$

and call this number the *Stanley depth* of M. For a squarefree module M we denote by

$$\operatorname{sqdepth}(M) = \max\{\operatorname{sdepth}(\mathscr{D}) \colon \mathscr{D} \text{ is a squarefree Stanley decomposition of } M\}$$

the *squarefree Stanley depth* of M. It is clear that $\operatorname{sqdepth}(M) \leq \operatorname{sdepth}(M)$. With the above notation Stanley's conjecture says that $\operatorname{depth}(M) \leq \operatorname{sdepth}(M)$.

It is known that the number of Stanley space of maximal dimension is independent of the special Stanley decomposition, see [20, 1018]. Apel [6] showed that if $I \subset S$ is a monomial ideal, then

$$\operatorname{sdepth}(S/I) \leq \min\{\dim(S/P) \colon P \in \operatorname{Ass}(S/I)\}.$$

The same result is true for any finitely generated $\mathbb{Z}^n$-graded S-module M. Indeed, let $\mathscr{D} = \bigoplus_{i=1}^t m_i K[Z_i]$ be a Stanley decomposition of M such that $\operatorname{sdepth}(\mathscr{D}) = \operatorname{sdepth}(M)$ and $P \in \operatorname{Ass}(M)$

an associated prime such that $\dim(S/P)=\min\{\dim(S/Q)\colon Q\in \operatorname{Ass}(M)\}$. Since $P\in\operatorname{Ass}(M)$, there exists a nonzero homogeneous element $m\in M$ such that $P=\operatorname{Ann}(m)$. On the other hand since $0\neq m\in M$, there exists a unique $1\leq k\leq t$ such that $m\in m_kK[Z_k]$. It is enough to show that $Z_k\cap P=\emptyset$. Let $m=m_kx^F$ for some $x^F\in K[Z_k]$. Suppose that $Z_k\cap P\neq\emptyset$, and choose $x_i\in Z_k\cap P$. Then $m_k(x^Fx_i)=mx_i=0$, a contradiction. This implies that $|Z_k|\leq\dim(S/P)$. In particular,

$$\operatorname{sdepth}(M)=\operatorname{sdepth}(\mathscr{D})\leq\dim(S/P)=\min\{\dim(S/Q)\colon Q\in\operatorname{Ass}(M)\}.$$

Let $I\subset S$ be a monomial ideal and $I=I_0\subset I_1\subset\cdots\subset I_r=S$ an $\mathbb{N}^n$-graded prime filtration of S/I with $I_i/I_{i-1}\cong S/P_{F_i}(-\mathbf{a}_i)$. It was shown in [12, page 398] that this prime filtration of S/I give us the Stanley decomposition $S/I=\bigoplus_{i=1}^r u_ik[Z_i]$ of S/I, where $Z_i=\{x_j\colon j\notin F_i\}$, and where $u_i=x^{\mathbf{a}_i}$. This Stanley decomposition is called the Stanley decomposition of S/I corresponding to the given prime filtration. With similar arguments one shows:

Proposition 3.1. *Let M be a finitely generated $\mathbb{Z}^n$-graded S-module. If $(0)=M_0\subset M_1\subset\cdots\subset M_r=M$ is a is a prime filtration of M such that $M_i/M_{i-1}\cong S/P_{F_i}(-\mathbf{a}_i)$, then $M\cong\bigoplus_{i=1}^r m_iK[Z_{F_i}]$ is a Stanley decomposition of M where $m_i\in M_i$ is a homogeneous element of degree $\mathbf{a}_i$ such that $(M_{i-1}:_S m_i)=P_{F_i}$ and $Z_{F_i}=\{x_j\colon j\notin F_i\}$.*

The following result is a generalization of [13, Lemma 3.1]. Again we omit the proof because the arguments are analogue to those in the proof of [13, Lemma 3.1].

Proposition 3.2. *Let M be a finitely generated $\mathbb{N}^n$-graded S-module. Then M has a squarefree Stanley decomposition if and only if M is a squarefree S- module.*

Remark 3.3. In [27, Proposition 2.5] Yanagawa proved that an $\mathbb{N}^n$-graded S-module M is squarefree if and only if there is a filtration of $\mathbb{N}^n$-graded submodules $0\subset M_1\subset\ldots\subset M_r=M$ of M such that each quotient $M_i/M_{i-1}\cong S/P_{F_i^c}(-F_i)$ for some $F_i\subset[n]$ where $F_i^c=[n]\setminus F_i$. Yanagawa's result and Proposition 3.1 implies one direction of Proposition 3.2.

As a generalization of [13, Theorem 3.3] we have the following. Again the same arguments like in the proof of [13, Theorem 3.3] work also here.

Theorem 3.4. *Let M be an $\mathbb{N}^n$-graded squarefree S-module. Then* $\operatorname{sqdepth}(M)=\operatorname{sdepth}(M)$.

Let $E=K\langle e_1,\ldots,e_n\rangle$ be the exterior algebra over an n-dimensional K-vector space V and N a finitely generated $\mathbb{N}^n$-graded E-module. Let $n\in N$ be a homogeneous element and $A\subset\{e_1,\ldots,e_n\}$. We set $\operatorname{supp}(n)=\operatorname{supp}(\deg(n))$ and $\operatorname{supp}(A)=\{j\colon e_j\in A\}$. We denote by $nK\langle A\rangle$ the the K-subspace of N generated by all homogeneous elements of the form ne_F, where $e_F\in K\langle A\rangle$. If the elements ne_F with $F\in\operatorname{supp}(A)$ form a K-basis of $nK\langle A\rangle$, then we call $nK\langle A\rangle$ a *Stanley space of dimension $|A|$*.

In case N is a squarefree and $nK\langle A\rangle\subset N$ is a Stanley space we have that $\operatorname{supp}(n)$ is squarefree and $\operatorname{supp}(n)\cap\operatorname{supp}(A)=\emptyset$. A direct sum $N=\bigoplus_{i=1}^t n_iK\langle A_i\rangle$ with Stanley spaces $n_iK\langle A_i\rangle$ is called a *Stanley decomposition* of N.

Proposition 3.5. *Let N be a squarefree E-module, and $N^\vee$ the E-dual of N. Then there exists a Stanley decomposition $N=\bigoplus_{i=1}^t n_iK\langle A_i\rangle$ of N if and only if there exists a Stanley decomposition $N^\vee=\bigoplus_{i=1}^t b_iK\langle A_i\rangle$ of $N^\vee$ with*

$$\operatorname{supp}(b_i)=[n]\setminus(\operatorname{supp}(A_i)\cup\operatorname{supp}(n_i)).$$

Proof. By Theorem 1.6 we have $N^\vee\cong N^*=\operatorname{Hom}_K(N,K(-\mathbf{1}))$. Hence we will show the assertion for N^*. Since $N=\bigoplus_{i=1}^t n_iK\langle A_i\rangle$, as an $\mathbb{N}^n$-graded K-vector space one has $N^*=\bigoplus_{i=1}^t(n_iK\langle A_i\rangle)^*$. Set $\operatorname{supp}(n_i)=F_i$ and $\operatorname{supp}(A_i)=G_i$. Then $F_i\cap G_i=\emptyset$ and the elements n_ie_H with $H\subseteq G_i$ form a K-basis of $n_iK\langle A_i\rangle$. Consequently, the dual elements $(n_ie_H)^*$ form a K-basis of $(n_iK\langle A_i\rangle)^*$.

Let $b_i = (n_i e_{G_i})^*$ and $H, L \subseteq G_i$. Then

$$(b_i e_H)(n_i e_L) = \pm b_i(n_i e_L e_H) = \begin{cases} 0, & \text{if} \quad L \neq G_i \setminus H, \\ \pm 1, & \text{if} \quad L = G_i \setminus H, \end{cases}$$

and for any $j \neq i$ and all $T \subset G_j$ one has $(b_i e_H)(n_j e_T) = \pm b_i(n_j e_T e_H) = 0$. This shows that $b_i e_H = \pm (n_i e_{G_i \setminus H})^*$ for any $H \subset G_i$. Therefore $(n_i K\langle A_i \rangle)^* = b_i K\langle A_i \rangle$ and $N^* = \bigoplus_{i=1}^t b_i K\langle A_i \rangle$. $\square$

Let M be a squarefree S-module and let N be its corresponding squarefree E-module. In Section 1 we showed that there is an isomorphism $\theta\colon M_{\mathrm{sq}} \to N$ of graded K-vector spaces. We will use this isomorphism to describe in the next lemma the relationship between squarefree Stanley decompositions of M and Stanley decompositions of N.

Lemma 3.6. (a) *Let $M = \bigoplus_{i=1}^t m_i K[Z_i]$ be a squarefree Stanley decomposition of M and*

$$A_i = \{e_j \colon j \in \mathrm{supp}(Z_i) \setminus \mathrm{supp}(m_i)\}.$$

Then $N = \bigoplus_{i=1}^t n_i K\langle A_i \rangle$ is a Stanley decomposition of N, where $n_i = \theta(m_i) \in N$ for $i = 1, \ldots, t$.
(b) *Conversely, if $N = \bigoplus_{i=1}^t n_i K\langle A_i \rangle$ is a Stanley decomposition of N and*

$$Z_i = \{x_j \colon j \in \mathrm{supp}(A_i) \cup \mathrm{supp}(n_i)\}.$$

Then $M = \bigoplus_{i=1}^t m_i K[Z_i]$ is a squarefree Stanley decomposition of M, where $m_i = \theta^{-1}(n_i) \in M$ for $i = 1, \ldots, t$.

Proof. (a): Since $M = \bigoplus_{i=1}^t m_i K[Z_i]$, one has

$$\bigcup_{i=1}^t \{m_i x_F : F \subset \mathrm{supp}(A_i)\}$$

forms a K-basis of M_{sq}, and hence

$$\bigcup_{i=1}^t \{\theta(m_i x_F) \colon F \subset \mathrm{supp}(A_i)\}$$

forms a K-basis of N. By Lemma 1.5 we have $\theta(m_i x_F) = (-1)^{\sigma(G_i,F)} n_i e_F$, where $G_i = \mathrm{supp}(m_i)$. Therefore

$$\bigcup_{i=1}^t \{n_i e_F : F \subset A_i\}$$

forms a K-basis of N.

(b): Let $x^{\mathbf{a}} \in K[Z_i]$. We can write $x^{\mathbf{a}} = x^{\mathbf{a}'} x^{\mathbf{b}}$ where $\mathbf{b} \in \mathbb{N}^n$ is a squarefree vector with $F = \mathrm{supp}(\mathbf{b}) \subset \mathrm{supp}(A_i)$. Then

$$m_i x^{\mathbf{a}} = (m_i x^{\mathbf{b}}) x^{\mathbf{a}'} = (-1)^{\sigma(G_i,F)} \theta^{-1}(n_i e_F) x^{\mathbf{a}'}.$$

Since $\theta^{-1}(n_i e_F) \neq 0$ and since M is squarefree and $\mathrm{supp}(\mathbf{a}') \subset \mathrm{supp}(\theta^{-1}(n_i e_F))$, one has $m_i x^{\mathbf{a}} \neq 0$. Therefore

$$\bigcup_{i=1}^t \{m_i x^{\mathbf{a}} \colon x^{\mathbf{a}} \in K[Z_i]\}$$

forms a K-basis of M. $\square$

Now we will present the main result of this section.

Theorem 3.7. *Let M be a squarefree S-module, and $M^\vee$ its Alexander dual. Then there exists a squarefree Stanley decomposition $M = \bigoplus_{i=1}^t m_i K[Z_i]$ of M if and only if there exists a squarefree Stanley decomposition $M^\vee = \bigoplus_{i=1}^t v_i K[W_i]$ of $M^\vee$ with $\operatorname{supp}(v_i) = [n] \setminus \operatorname{supp}(Z_i)$ and $W_i = \{x_j \colon j \in [n] \setminus \operatorname{supp}(m_i)\}$.*

Proof. Let $M = \bigoplus_{i=1}^t m_i K[Z_i]$ be a squarefree Stanley decomposition of M. If we set $F_i = \operatorname{supp}(m_i)$ and $G_i = \operatorname{supp}(Z_i) \setminus F_i$, then $F_i \cap G_i = \emptyset$. Let N be the squarefree E-module corresponding to M. Then by Lemma 3.6(a), N has a Stanley decomposition

$$N = \bigoplus_{i=1}^t n_i K\langle A_i \rangle$$

where $n_i = \theta(m_i)$ and $G_i = \operatorname{supp}(A_i)$. Hence by Proposition 3.5, $N^\vee$ has a decomposition $N^\vee = \bigoplus_{i=1}^t b_i K\langle A_i \rangle$ with $\operatorname{supp}(b_i) = [n] \setminus (G_i \cup F_i)$. Therefore by Lemma 3.6(b), $M^\vee$ the corresponding squarefree S-module to $N^\vee$ has a decomposition as required. $\square$

Associated to any finitely generated $\mathbb{N}^n$-graded S-module M is a *minimal free $\mathbb{Z}^n$-graded resolution*

$$0 \to \bigoplus_j S(-\mathbf{a}_j)^{\beta_{r,j}(M)} \to \cdots \to \bigoplus_j S(-\mathbf{a}_j)^{\beta_{1,j}(M)} \to \bigoplus_j S(-\mathbf{a}_j)^{\beta_{0,j}(M)} \to 0$$

where $S(-\mathbf{a}_j)$ denote the $\mathbb{Z}^n$-graded S-module obtained by shifting the degrees of S by $\mathbf{a}_j$. The number $\beta_{i,j}(M)$ is the *ij-th graded Betti number* of M. The regularity of M is

$$\operatorname{reg}(M) = \max\{|\mathbf{a}_j| - i \colon \text{for all } i,j\}.$$

Let M be a squarefree $\mathbb{N}^n$-graded S-module. If Stanley's conjecture holds for M, then by Theorem 3.4 we may assume that there exists a squarefree Stanley decomposition $M = \bigoplus_{i=1}^t m_i K[Z_i]$ of M such that $|Z_i| \geq \operatorname{depth}(M)$. Also by Theorem 3.7 there exists a squarefree Stanley decomposition $M^\vee = \bigoplus_{i=1}^t v_i K[W_i]$ of the Alexander dual of M such that $|\deg(v_i)| = n - |Z_i| \leq n - \operatorname{depth}(M)$. On the other hand $\operatorname{proj\,dim}(M) = \operatorname{reg}(M^\vee)$, see [25, Corollary 3.7]. Since $\operatorname{depth}(M) + \operatorname{proj\,dim}(M) = n$, see [7, Theorem 1.3.3], we have $|\deg(v_i)| \leq \operatorname{reg}(M^\vee)$ for all i. Therefore we will get the following:

Corollary 3.8. *Let M be a squarefree $\mathbb{N}^n$-graded S-module and $M^\vee$ its Alexander dual. Then Stanley's conjecture holds for M if and only if $M^\vee$ has a squarefree Stanley decomposition $M^\vee = \bigoplus_{i=1}^t v_i K[W_i]$ with $|\deg(v_i)| \leq \operatorname{reg}(M^\vee)$ for all i.*

In the case that $I \subset S$ is a monomial ideal and $M = S/I$ or $M = I$, then we may consider the standard grading for S and M by setting $\deg(x_i) = 1$ for $i = 1, \ldots, n$. In this case a minimal graded free resolution of I is

$$0 \to \bigoplus_j S(-j)^{\beta_{r,j}(M)} \to \cdots \to \bigoplus_j S(-j)^{\beta_{1,j}(M)} \to \bigoplus_j S(-j)^{\beta_{0,j}(M)} \to I \to 0.$$

Suppose that all monomial minimal generators of I are of degree d. Then I has a *linear resolution* if for all $i \geq 0$, $\beta_{i,j} = 0$ for all $j \neq i + d$.

Let $F \subset G \subset [n]$. We denote the interval $\{H \colon F \subseteq H \subseteq G\}$ by $[F,G]$. A partition $\mathbf{P} \colon \Delta = \bigcup_{i=1}^t [F_i, G_i]$ of Δ is a disjoint union of intervals of Δ. A simplicial complex Δ is called partitionable if there is a partition $\mathbf{P} \colon \Delta = \bigcup_{i=1}^t [F_i, G_i]$ of Δ such that $\{G_1, \ldots, G_t\}$ is the set of facets of Δ. In [22] Stanley conjectured that any Cohen-Macaulay simplicial complex is partitionable, see also [23]. In [13, Corollary 3.5] it was shown that this conjecture is a special case of Stanley's conjecture on Stanley decompositions. Indeed, the authors proved that if $\mathbf{P} \colon \Delta = \bigcup_{i=1}^t [F_i, G_i]$ is a partition of Δ, then $\mathscr{D}(\mathbf{P}) \colon S/I_\Delta = \bigoplus x_{F_i} K[Z_{G_i}]$ is a squarefree Stanley decomposition of S/I_Δ, where $x_{F_i} = \prod_{j \in F_i} x_j$ and $Z_{G_i} = \{x_j \colon j \in G_i\}$. Hence we get the following corollary.

Corollary 3.9. *A Cohen-Macaulay simplicial complex Δ is partitionable if and only if $I_{\Delta^\vee}$ has a squarefree Stanley decomposition $I_{\Delta^\vee} = \bigoplus_{i=1}^t u_i K[Z_i]$ such that $\{u_i, \dots, u_t\} = G(I_{\Delta^\vee})$.*

Proof. By Eagon-Reiner [8] Δ is Cohen-Macaulay if and only if $I_{\Delta^\vee}$ has a linear resolution. Also by a result of Terai [25] $\operatorname{proj\,dim}(S/I_\Delta) = \operatorname{reg}(I_{\Delta^\vee})$ for any simplicial complex Δ.

On the other hand by Corollary 3.8 the Cohen-Macaulay simplicial complex Δ is partitionable if and only if $I_{\Delta^\vee}$ has a squarefree Stanley decomposition $I_{\Delta^\vee} = \bigoplus_{i=1}^t u_i K[Z_i]$ such that $\deg u_i \leq \operatorname{reg}(I_{\Delta^\vee}) = d$, where d is the degree of any minimal monomial generator of $I_{\Delta^\vee}$. Since $u_i \in I_{\Delta^\vee}$, one has $\deg(u_i) \geq d$ for all i. This shows that $u_i \in G(I_{\Delta^\vee})$ and hence $\{u_i, \dots, u_t\} \subset G(I_{\Delta^\vee})$. The other inclusion is obvious. $\square$

Corollary 3.9 shows that Stanley's conjecture which says that any Cohen-Macaulay simplicial complex is patitionable is equivalent to say that any squarefree monomial ideal $I \subset S$ which has a linear resolution has a Stanley decomposition $I = \bigoplus_{i=1}^t u_i K[Z_i]$ such that $\{u_1, \dots, u_t\} = G(I)$.

This results lead us to make the following conjecture which in the case of squarefree $\mathbb{N}^n$-graded S-module is equivalent to Stanley's conjecture on Stanley decompositions.

Conjecture 3.10. *Let $S = K[x_1, \dots, x_n]$, and let M be a finitely generated $\mathbb{Z}^n$-graded S-module. Then there exists a Stanley decomposition*

$$M = \bigoplus_{i=1}^t m_i K[Z_i],$$

of M, where $|m_i| \leq \operatorname{reg} M$ for all i.

Let $\mathscr{D}$ be a Stanley decomposition of M. We call the maximal $|m_i|$ in $\mathscr{D}$ the *h-regularity* of $\mathscr{D}$, and denote it by $\operatorname{hreg}(\mathscr{D})$. Maclagan and Smith [16, Remark 4.2] proved that $\operatorname{hreg}(\mathscr{D}) \geq \operatorname{reg}(M)$ in the case that $M = S/I$, where I is a monomial ideal, and $\mathscr{D}$ is a Stanley filtration. We set $\operatorname{hreg}(M) = \min\{\operatorname{hreg}(\mathscr{D})\colon \mathscr{D}$ is a Stanley decomposition of $M\}$, and call this number the *h-regularity* of M. With the notation introduced our conjecture says that $\operatorname{hreg}(M) \leq \operatorname{reg}(M)$.

Let M be a finitely generated $\mathbb{N}^n$-graded S-module which is generated by homogeneous elements $n_1, \dots, n_s$. It is clear that $|n_i| \leq \operatorname{reg}(M)$ for $i = 1, \dots, s$. We want to show that $|n_i| \leq \operatorname{hreg}(M)$ for $i = 1, \dots, s$. Let $\mathscr{D} = \bigoplus_{i=1}^t m_i K[Z_i]$ be a Stanley decomposition of M such that $\operatorname{hreg}(\mathscr{D}) = \operatorname{hreg}(M)$, and $|n_r| = \max\{|n_i|\colon i = 1, \dots, s\}$. Since $n_r \in M$ is a homogeneous element, there exists a $j \in [t]$ such that $n_r \in m_j K[Z_j]$. On the other hand $m_j \in M$ and n_r is a generator. Therefore we have $m_j = n_r$ and $|n_r| = |m_j| \leq \operatorname{hreg}(\mathscr{D})$.

Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal. Apel [5] proved that if $\operatorname{depth}(I) \leq 2$ or $n \leq 3$, then Stanley's conjecture holds for I. Also if $n \leq 3$, then Stanley's conjecture holds for S/I, see [6] or [20]. Furthermore in [13] the authors showed that Stanley' conjecture holds for S/I if I is a complete intersection, S/I is Cohen–Macaulay of codimension 2, or S/I is Gorenstein of codimension 3. If $I = I_\Delta$ is a squarefree monomial ideal, then $\operatorname{proj\,dim}(I_\Delta) = \operatorname{reg}(S/I_{\Delta^\vee})$. The discussions above together with Corollary 3.8 yield the following:

Corollary 3.11. *Let $I \subset S = K[x_1, \dots, x_n]$ be a squarefree monomial ideal. Then*

(i) *Conjecture 3.10 holds for I and for S/I if $n \leq 3$;*
(ii) *Conjecture 3.10 holds for S/I if $\operatorname{reg}(S/I) \geq n-2$;*
(iii) *Conjecture 3.10 holds for I if $\operatorname{reg}(I) = 2$.*

Let $I = (u_1, \dots, u_m)$ be a monomial ideal in S. According to [14], the monomial ideal I has linear quotients if one can order the set of minimal generators of I, $G(I) = \{u_1, \dots, u_m\}$, such that the ideal $(u_1, \dots, u_{i-1}) : u_i$ is generated by a subset of the variables for $i = 2, \dots, m$.

Assume that $I=(u_1,\dots,u_m)$ is a monomial ideal which has linear quotients with respect to the given order. Set $I_i=(u_1,\dots,u_{i-1}):u_i$, $Z_i=X\setminus G(I_i)$ for $i=2,\dots,m$ and $Z_1=X$. We denote $r_i=|G(I_i)|$ for $i=2,\dots,m$ and $r(I)=\max\{r_i:\ i=2,\dots,s\}$. By [10, page 539] $\operatorname{depth}(I)=n-r(I)$.

Corollary 3.12. *Let $I\subset S$ be a monomial ideal with linear quotients. Then Stanley's conjecture on Stanley decompositions holds for I.*

Proof. Suppose $I=(u_1,\dots,u_m)$ has linear quotients with respect to the given order. Then $\mathscr{G}$: $(0)\subset J_1=(u_1)\subset\ldots\subset J_{m-1}=(u_1,\dots,u_{m-1})\subset J_m=I$ is a prime filtration of I. Hence by Proposition 3.1 $\mathscr{D}=\bigoplus_{i=1}^{s}u_iK[Z_i]$ is a Stanley decomposition of I with $\operatorname{sdepth}(\mathscr{D})=n-r(I)=\operatorname{depth}(I)$. $\square$

In the decomposition above of I, all u_i are the minimal monomial generators of I. Therefore we have

Corollary 3.13. *If $I\subset S$ is a monomial ideal which has linear quotient, then Conjecture 3.10 holds for I.*

In [11] it was shown that if I is monomial ideal with 2-linear resolution, then I has linear quotients. Therefore Stanley's conjecture on Stanley decompositions and Conjecture 3.10 holds for such monomial ideals.

REFERENCES

[1] A. Aramova, L. L. Avramov, J. Herzog, Resolutions of monomial ideals and cohomology over exterior algebras. Trans. Amer. Soc. **352** (2000), 579–594.

[2] A. Aramova, J. Herzog, Almost regular sequences and Betti numbers. Amer. J. Math. **122** (2000), 689–719.

[3] S. Ahmad, D. Popescu, Sequentially Cohen-Macaulay monomial ideals of embedding dimension four. math.AC/0702569.

[4] I. Anwar, D. Popescu, Stanley Conjecture in small embedding dimension. math.AC/0702728.

[5] J. Apel, On a conjecture of R. P. Stanley. Part I-Monomial Ideals, J. of Alg. Comb. **17** (2003), 36–59.

[6] J. Apel, On a conjecture of R. P. Stanley. Part II-Quotients Modulo Monomial Ideals, J. of Alg. Comb. **17** (2003), 57–74.

[7] W. Bruns and J. Herzog, *Cohen-Macaulay rings*. Cambridge University Press, Cambridge, 1993.

[8] J. Eagon and V. Reiner, Resolutions of Stanley–Reisner rings and Alexander duality. J. Pure Appl. Algebra, **130** (1998), 265–275.

[9] J. Herzog, T. Hibi, *Monomials*. Book in preparation.

[10] J. Herzog, T. Hibi, The depth of powers of an ideal. J. Algebra, **291** (2005), 534–550.

[11] J. Herzog, T. Hibi, X. Zheng, Monomial ideals whose powers have a linear resolution. Math. Scand. **95** (2004), no. 1, 23–32

[12] J. Herzog, D. Popescu, Finite filtrations of modules and shellable multicomplexes. Manuscripta Math. **121** (2006), no. 3, 385–410.

[13] J. Herzog, A. Soleyman Jahan, S. Yassemi, Stanley decompositions and partitionable simplicial complexes. to appear in J. Algebraic Combin. .

[14] J. Herzog, Y. Takayama, Resolutions by mapping cones. The Roos Festschrift volume, 2. Homology Homotopy Appl. **4** (2002), no. 2, part 2, 277–294.

[15] H. Masumura, *Commutative Ring theory*. Cambridge, 1986.

[16] D. Maclagan, G. G. Smith, Uniform bounds on multigraded regularity. J. Algebraic Geometry, **14** (2005), 137–164.

[17] U. Nagel, T. Römer, Glicci simplicial complexes. math.AC/0704.3283

[18] T. Römer, Generalized Alexander duality and applications. Osaka J. Math. **38** (2001), no. 2, 469–485.

[19] T. Römer, Cohen-Macaulayness and squarefree modules. Manuscripta Math. **104** (2001), no. 1, 39–48.

[20] A. Soleyman Jahan, Prime filtrations of monomial ideals and polarizations. J. Algebra, **312** (2007), 1011–1032.

[21] R. P. Stanley, Linear Diophantine equations and local cohomology. Invent. Math. **68**, (1982), 175–193.

[22] R. P. Stanley, *Combinatorics and Commutative Algebra*. Birkhäuser, 1983.

[23] R. P. Stanley, Positivity Problems and Conjectures in Algebraic Combinatorics, In Mathematics: Frontiers and Perspectives (V. Arnold, M. Atiyah, P. Lax, and B. Mazur, eds.), American Mathematical Society, Providence, RI, 2000, pp. 295-319.

[24] A. Soleyman Jahan, X. Zheng, Pretty clean monomial ideals and linear quotients. math.AC/0707.2914.

[25] N. Terai, Generalization of Eagon–Reiner theorem and h-vectors of graded rings. Preprint 2000.

[26] R. H. Villarreal, *Monomial Algebras*, Dekker, NewYork, NY, 2001.

[27] K. Yanagawa, Alexander duality for Stanley-Reisner rings and squarefree $\mathbb{N}^n$-graded modules. J. Algebra, **225**, (2000), 630–645.

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